# Math Circles - Pigeonhole Principle - Fall 2022

## Exercises

1. For Halloween this year, you decide to adopt a new strategy to give out candy to the trick-or-treaters. The night before Halloween, you put together packages of candy; one package for each of the 42 trick-or-treaters that you expect, plus one for yourself (because of course you want candy too). In order to be nice, you make sure that every package has at least one candy in it, but after that, you randomly distribute the candy amongst the packages. Once the candy has all been distributed, you choose the biggest package and keep it for yourself, and leave the other packages to the trick-or-treaters. If candy comes in boxes of 45 pieces, how many boxes of candy do you need to buy to ensure you end up with at least 7 pieces of candy in your package?

   *Solution.* Since there are 42 packages for the trick-or-treaters and one package for yourself, there are 43 packages total. We know that each of these packages will have one piece of candy; we can consider these 43 pieces separately from the others.

   For the rest of the candy, let the packages be our holes and let the candy be our pigeons. This gives us 43 holes, and we want one of these holes to have at least 6 pigeons in it (since we already put one piece of candy in each package). By the generalized pigeonhole principle, we will need at least $43 \cdot 5 + 1 = 215$ pieces of candy.

   So, in total, we will need $215 + 43 = 258$ pieces of candy. Since candy comes in boxes of 45, we will need 6 boxes of candy. ■

2. Prove that, on an $8 \times 8$ chessboard, it is impossible to place nine rooks[1] so that no two rooks threaten each other.

   *Solution.* Suppose we have nine rooks, and wish to place them on an $8 \times 8$ chessboard. In the pigeonhole principle, let the 8 rows of the chessboard be our holes and let the rooks be our pigeons. Then, by the pigeonhole principle, no matter how we place the rooks on the board, we will end up with at least two rooks in the same row. But, since rooks move along rows and columns, the two rooks on the same row will be threatening each other. ■

3. Imagine that you're trying to cover an $8 \times 8$ chessboard with dominoes, where each domino covers two adjacent squares. It is easy to see that this is possible on a normal chessboard, however is it still possible if you remove two diagonally-opposite corners?

   *Solution.* No. Recall that on a chessboard, the squares are coloured with two colours in an alternating pattern, such that two adjacent squares are coloured different colours. In total, we have 64 squares, so we have 32 of each colour. Notice:

   - Each domino covers two squares of opposite colour.
   - Both squares in a pair of diagonally-opposite squares are the same colour.

[1] Recall that rooks can move as far as they want along the column or row that they are in.

Without loss of generality, assume that that the corners we removed were white. So, we have $(64-2)/2 = 31$ dominoes to work with, and 32 black squares, which must each be covered by one of the dominoes. By the pigeonhole principle, this means that at least one domino will need to cover two black squares. However, this is impossible, since each domino must cover two squares of opposite colour. So, it is not possible to cover the chessboard with dominoes in this way. ∎

4. Suppose we have five distinct lattice points[2] on the $xy$-plane. If each pair of points is connected by a line, show that at least one of these lines has a lattice point in its interior.[3]

   *Solution.* Suppose we have five distinct lattice points. Since both coordinates of each of point are integers, then each point must be of one of the following four forms:

$$(\text{even}, \text{even})$$
$$(\text{even}, \text{odd})$$
$$(\text{odd}, \text{even})$$
$$(\text{odd}, \text{odd})$$

   By the pigeonhole principle, at least two of our lattice points must have the same form. Call these points $(x_1, y_1)$ and $(x_2, y_2)$.

   Notice that, since

$$\text{even} + \text{even} = \text{even}$$

   and

$$\text{odd} + \text{odd} = \text{even}$$

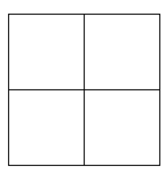
   we must have that

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (\text{even}, \text{even}).$$

   So, $\frac{x_1+x_2}{2}$ and $\frac{y_1+y_2}{2}$ are both integers, and hence $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is a lattice point. But $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is the midpoint of the line segment from $(x_1, y_1)$ to $(x_2, y_2)$, and since $(x_1, y_1)$ and $(x_2, y_2)$ are distinct, $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is in the interior of the line segment, as desired. ∎

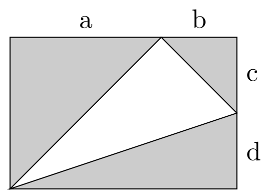
5. Given nine points inside a unit square,[4] show that three of these points must form a triangle whose area is less than or equal to $\frac{1}{8}$.

   *Solution.* Divide the square into 4 identical $\frac{1}{2} \times \frac{1}{2}$ quadrants, as shown:



   By the generalized pigeonhole principle, one of these quadrants must contain three points. Consider the triangle formed by these points, and inscribe it in a rectangle such that one vertex lies in a corner of the rectangle, and the other two vertices each lie one of the opposite sides, as shown:

[2]A *lattice point* is a point whose coordinates are integers.
[3]The *interior* of a line segment means all points of the line, except for its endpoints.
[4]A *unit square* is a square with side length 1.




The area of the rectangle is

$$(a+b)(c+d) = ac + ad + bc + bd$$

and since the rectangle is completely contained within a square of area $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, we get that

$$ac + ad + bc + bd \leq \frac{1}{4}.$$

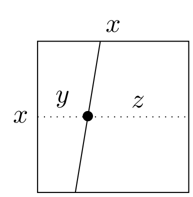
The area of the triangle is the area of the rectangle, minus the shaded area, which is

$$\begin{aligned} A =&(a+b)(c+d) - \frac{1}{2}a(c+d) - \frac{1}{2}d(a+b) - \frac{1}{2}bc \\ =&\frac{1}{2}ac + \frac{1}{2}bc + \frac{1}{2}bd \\ \leq&\frac{1}{2}(ac+ad+bc+bd) \\ \leq&\frac{1}{2} \cdot \frac{1}{4} \\ =&\frac{1}{8} \end{aligned}$$

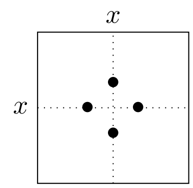
So, there must exist a triangle of area less than or equal to $\frac{1}{8}$. ■

6. We are given a square that has 9 lines drawn on it. Each line divides the square into two quadrilaterals such that one quadrilateral contains $\frac{1}{3}$ of the area of the square and the other quadrilateral contains $\frac{2}{3}$ of the area of the square. Prove that at least 3 of these 9 lines pass through the same point.

   *Solution.* Suppose the square has sides of length $x$, and consider a line $L$ as described in the question. Since $L$ divides the square into two quadrilaterals, it must be the case that its endpoints are on opposite sides of the square. So, it looks something like this:


   where $y$ and $z$ are the distances between the sides of the square and the midpoint of the diagonal line. Notice that the area of the trapezoid on the left is given by $xy$ and the area of the trapezoid on the right is $xz$. So, without loss of generality, we have that $xy = \frac{1}{3}x^2$ and $xz = \frac{2}{3}x^2$, and hence that $y = \frac{1}{3}$ and $z = \frac{2}{3}$. So, the midpoint of each line must be $\frac{1}{3}$ of the distance from one of the sides of the square. In other words, each of the nine lines must pass through one of the following points:




If we let the points be our holes and the lines be our pigeons, then by the generalized pigeonhole principle, at least 3 of the lines must pass through the same point. ■